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FRACTURE MECHANICS OF MULTILAYERED SHELLS. THEORY OF DELAMINATION CRACKS *

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Delamination cracks are considered, which develop along the boundaries of different layers in a multilayered thin shell. The characteristic linear dimension of the crack in planform is assumed to be large compared with the shell thickness. A general theory suitable for materials with any inelastic properties is based on an additional boundary condition on the moving contour of the crack, which is derived by using a heuristic hypothesis. The theory of invariant Γ-integrals and the general theory of fracture are also utilized. Model experiments are indicated which enable fracture diagrams, needed for carrying the theoretical computations out to numbers, by test means to be determined. As an illustration of the general theory, a one-dimensional problem on the fracture of a two-layer beam from ideally elastic-plastic materials is studied in detail. Furthermore, the following questions are examined: the subcritical growth of delamination cracks in multilayered shells from elastic-plastic materials, the dependence of limit loads on the loading path, and delamination fatigue cracks. An exact solution is given for the problem of elliptic, parabolic, and hyperbolic cracks in a plane two-layered plate, an axisymmetric delamination crack in a two-layered cylindrical shell, and an elliptical delamination crack in a two-layered plane membrane.

1. **Boundary conditions on the contour of a delamination crack.** Consider a multilayer thin shell with a delamination crack developing along the boundaries of the different layers. In planform, the characteristic linear dimension of the crack will be assumed to be large compared with the shell thickness.

Let S denote the curvilinear surface of the delamination crack, and L the contour of this surface (Fig.1). The crack divides the initial whole shell into two separate shells in the domain S: S^+ and S^-. We shall give the superscripts plus and minus to all quantities referring to these shells. It is required to establish how the contour changes in time and as a function of the external loads. This is a non-linear problem of shell theory since the contour L is not known in advance, and it must be determined during the solution.

Let us set up all the remaining boundary conditions for this problem. Three shells are in conjunction along the contour L, S^+, S^- (in the domain S) and the initial shell S^0 (outside the domain S). Each satisfies the appropriate differential equations of the theory of thin shells.

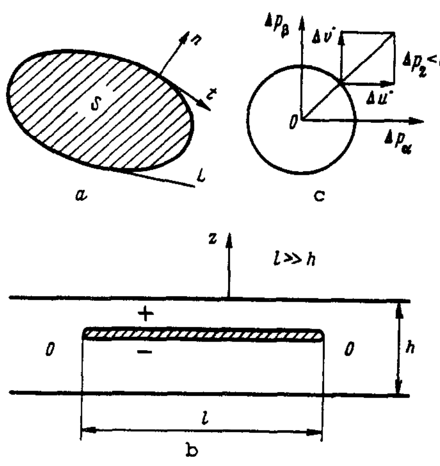

Fig. 1

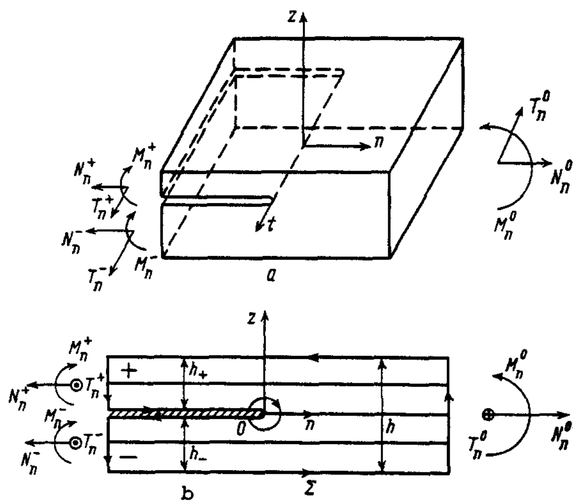

Fig.2

*Prikl.Matem.Mekhan.,47,5,832-845,1983

The following boundary conditions should be satisfied on the juncture contour L (the superscript zero is ascribed to quantities referring to the shell S^0):

The continuity conditions

$$u^+ = u^- = u^\circ,\ v^+ = v^- = v^\circ,\ w^+ = w^- = w^\circ \tag{1.1}$$

The equilibrium equations

$$N_n^\circ = N_n^+ + N_n^-, \quad T_n^\circ = T_n^+ + T_n^- \tag{1.2}$$

$$M_n^\circ = M_n^+ + M_n^- \tag{1.3}$$

$$Q_n^\circ = Q_n^+ + Q_n^-, \quad M_{nt}^\circ = M_{nt}^+ + M_{nt}^- \tag{1.4}$$

Here t and n are the tangent and normal to the contour L on the neutral surface, u, v, w are the displacement vector components of the neutral shell surface along the α, β and z axes, respectively, $\alpha\beta$ is an orthogonal curvilinear Gaussian coordinate system on the neutral shell surface, z is the normal to this surface, N and T are the normal and tangential forces, M_n is the bending moment with bending axis n, M_{nt} is the torque in the nt plane, and Q is the transverse force.

A delamination crack can be open or closed. In the latter case, the following conditions of joint operation of the shells S^+ and S^- must be satisfied at each point of the domain S:

The normal continuity condition

$$w^+ = w^- \tag{1.5}$$

The equation of ultimate equilibrium (in the slip state)

$$\Delta p_\alpha^2 + \Delta p_\beta^2 = f(\Delta p_z) \tag{1.6}$$

and the equation of the associated plasticity law in the slip state (see Fig. 1c)

$$\Delta u^\cdot/\Delta v^\cdot = \Delta p_\alpha/\Delta p_\beta;\ \Delta u = u^+ - u^-,\ \Delta v = v^+ - v^- \tag{1.7}$$

$$\Delta p_\alpha^+ = -\Delta p_\alpha^- = \Delta p_\alpha, \quad \Delta p_\beta^+ = -\Delta p_\beta^- = \Delta p_\beta,$$

$$\Delta p_z^+ = -\Delta p_z^- = \Delta p_z$$

Here Δp_α, Δp_β, and Δp_z denote the appropriate intensity vector components of the additional distributed load that occurs because of mutual superposition of the shells S^+ and S^-; in the simplest case the function f satisfies the Coulomb dry friction law $f = \sqrt{k_0 + \Delta p_z \operatorname{tg} \rho}$ (k_0 and ρ are the adhesion coefficient and the angle of friction), and $\Delta u^\cdot$ and $\Delta v^\cdot$ are the velocity components of the mutual slip of the shells.

Therefore, in the case of overlying shells, three new unknown functions Δp_α, Δp_β and Δp_z, dependent on α and β and participating in the equilibrium equations of each shell, appear in the domain S. The three equations (1.5)-(1.7) together with the equations of thin elastic shell theory comprise a closed system which naturally turns out to be non-linear because of the non-linearity of (1.6) and (1.7). The functions Δp_α and Δp_β should vanish on the contour L. Note that if the contact friction forces are neglected and we put $\Delta p_\alpha = \Delta p_\beta = 0$, then (1.6) and (1.7) vanish; the problem again becomes linear while one new equation (1.5) will correspond to one new function Δp_z.

When there is partial contact between the shells S^+ and S^-, the contact domain itself should be determined when solving the problem. On the boundary separating the open and closed delamination domains, obvious connecting conditions should be satisfied for each of the shells.

All the equations formulated enable problems to be set up and solved when the position of the crack front is known. To deduce additional boundary conditions on the delamination crack front, we will make use of invariant Γ-integrals and the general theory of fracture proposed in /1/.

Let us consider the boundary layer zone in the neighbourhood of a certain point O of the contour L of the delamination crack (Fig.2). It is assumed that the width of this zone (along the normal n to the contour) is small compared to the characteristic linear dimension of the crack in planform (in particular, to the radius of curvature of the contour L and the radius of curvature of the shell) but large compared with the shell thickness (in practice, two or three times greater than the shell thickness on the basis of exact solutions). The shell-theory approximation is not suitable in the boundary layer and an exact three-dimensional theory must be relied upon. In the ordinary thin shell approximation the elastic field in the boundary layer can be considered planar in the neighbourhood of the point O, i.e., independent of the coordinate t along the crack contour, but the shell itself can be regarded as a plane multilayer plate with a crack front along the t axis of rectangular Cartesian coordinates $Otnz$ (Fig.2).

We introduce the following assumption (it is heuristic in nature, results from intuitive physical considerations, and is confirmed by well-known exact solutions).

Delamination crack development at the point O is determined by the following quantities at this point: $N_n^+, N_n^-, N_n^0, T_n^+, T_n^-, T_n^0, M_n^+, M_n^-, M_n^0$ *and is independent of the other bending* *moments and torques and the transverse forces at this point.* Crack development at the point O is also independent of the presence of distributed tangential and normal loads at the crack edges and the side surfaces of the shell near the point O.

This assumption will be satisfied more exactly, the smaller the ratios h/l, h/R, where h is the shell thickness, R is the shell radius of curvature, and l is the characteristic linear dimension of the crack in planform. On this assumption, the crack development at the point O can be studied in a plane boundary layer quadrant (Fig. 2b) with just the above-mentioned loads taken into account and the external loads on the crack edges neglected. We hence have the following boundary value problem in the plane nz for a multilayer strip with a crack:

$$\left.\begin{array}{l} z = h_+,\ z = -h_- \\ z = 0,\ n < 0 \end{array}\right\},\quad \sigma_z = \tau_{zn} = \tau_{zt} = 0 \tag{1.8}$$

$$\begin{array}{l} n \to \infty,\ \varepsilon_n = \varepsilon_n^\circ + \varkappa_n^\circ z,\ \gamma_{nt} = \gamma_{nt}^\circ \\ n \to -\infty\ (z > 0),\ \varepsilon_n = \varepsilon_n^+ + \varkappa_n^+ z,\ \gamma_{nt} = \gamma_{nt}^+ \\ n \to -\infty\ (z < 0),\ \varepsilon_n = \varepsilon_n^- + \varkappa_n^- z,\ \gamma_{nt} = \gamma_{nt}^- \end{array} \tag{1.9}$$

Here ε_n and γ_{nt} are the normal and tangential strains. Strict adhesion conditions hold along the interface of the layers for $z = \text{const}$.

In the case of linearly elastic bodies, the following relations hold

$$\begin{array}{l} N_n^\circ = k_e^\circ \varepsilon_n^\circ,\quad N_n^+ = k_e^+ \varepsilon_n^+,\quad N_n^- = k_e^- \varepsilon_n^- \\ T_n^\circ = k_s^\circ \gamma_{nt}^\circ,\quad T_n^+ = k_s^+ \gamma_{nt}^+,\quad T_n^- = k_s^- \gamma_{nt}^- \\ M_n^\circ = k_b^\circ \varkappa_n^\circ,\quad M_n^+ = k_b^+ \varkappa_n^+,\quad M_n^- = k_b^- \varkappa_n^- \\ \left(k_e = \sum_i \frac{h_i E_i}{1 - \nu_i^2},\quad k_s = \sum_i h_i G_i,\quad k_b = \sum_i \frac{E_i I_i}{1 - \nu_i^2}\right) \end{array}$$

Here k_e, k_s, k_b are the tensile, shear, and bending stiffnesses of the multilayer strip, respectively. They are obtained by summing the appropriate stiffnesses of all the layers in the strip; $h_i, E_i, G_i, \nu_i, I_i$ are the corresponding i-th layer parameters: thickness, Young's modulus, shear modulus, Poisson's ratio, and moment of intertia with respect to the neutral axis of the strip.

Let us examine the closed contour Σ in the Onz plane formed by the opposite edges of the crack, a circle of quite small radius enclosing the point O, the side planes of the strip, and its transverse sections as $n \to \pm\infty$ (Fig. 2b). According to the theory of invariant Γ-integrals, the following equation holds /2/:

$$\int_\Sigma (U n_1 - \sigma_{ij} n_j u_{i,1})\, d\Sigma = 0 \quad (i = 1, 2, 3;\ j = 1, 2) \tag{1.10}$$

Here n_j is the unit normal to the contour Σ, σ_{ij} is the stress, u_i the displacement, and U is the elastic potential per unit volume. Equation (1.10) is also valid for any inelastic bodies (elastic-plastic, viscoelastic, etc.) for quasistationary motion of the point O along the axis n at a velocity considerably less than the speed of sound in the strip; here, U is understood to be the specific strain energy.

Let Γ denote the magnitude of the integral in (1.10) around the circle enclosing the point O (Γ is the residue at the point O). On the basis of the boundary conditions (1.8) and (1.9), we have from (1.10)

$$\begin{array}{l} \Gamma = \int_L (U - \sigma_{i1} u_{i,1})\, dz = \int_L (U - \sigma_n \varepsilon_n - \tau_{nz} u_{z,n} - \tau_{nt} \gamma_{nt})\, dz = \\ \int_L U\, dz - \varepsilon_n \int_L \sigma_n\, dz - \varkappa_n \int_L z\sigma_n\, dz - u_{z,n} \int_L \tau_{nz}\, dz - \\ \gamma_{nt} \int_L \tau_{nt}\, dz = \int_L U\, dz - (\varepsilon_n N_n + \varkappa_n M_n + \gamma_{nt} T_n)|_L = [U_s];\quad \int_L = \int\limits_{\substack{-h_- \\ n \to -\infty}}^{+h_+} - \int\limits_{\substack{-h_- \\ n \to +\infty}}^{+h_+} \\ [U_s] = -U_s^\circ + U_s^+ + U_s^-;\quad U_s = -U_d + N_n \varepsilon_n + M_n \varkappa_n + T_n \gamma_{nt} \\ [U_d] = \int_L U\, dz;\quad U_d = \int (N_n\, d\varepsilon_n + M_n\, d\varkappa_n + T_n\, d\gamma_{nt}) \end{array} \tag{1.11}$$

Here U_d is the strain energy per unit area of the multilayer plate in planform, U_s is the additional strain energy (the physical meaning of U_d and U_s is clear from the one-dimensional tension diagram of Fig.3 obtained as follows: we stretch the specimens by strains $\varepsilon_n = \varepsilon_n(t)$ given at the time t, and take off the corresponding force $N_n = N_n(t)$; then by eliminating t we construct the curve $N_n = N_n(\varepsilon_n)$ for a given loading path). The square brackets denote a jump in the quantity enclosed in the brackets when intersecting the crack front.

The equation on the crack front

$$\Gamma = [U_s] \tag{1.12}$$

is the main result of delamination crack theory in multilayer shells for arbitrary materials. It follows, for instance, that if the rate of crack growth V is much greater than h/τ, where τ is the characteristic time of progress of the after-effect reaction of the multilayer material, then the crack development will be determined just by the instantaneous reaction of the material as a whole (i.e., by its elastic-plastic properties) and will be independent of the after-effect reaction of the material (i.e., of its creep and viscosity properties).

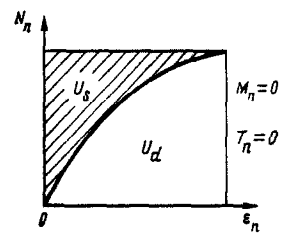

Fig.3

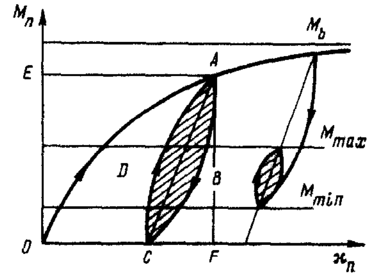

Fig.4

Indeed, the boundary layer width Δn practically equals $(2-3)\,h$ (the asymptotic infinity, Fig.2), hence for $V \gg h/\tau$ the right side of (1.12) depends only on the instantaneous reaction of the multilayer material. The left side of (1.12), the quantity Γ, is determined by the properties of the material in a very small neighbourhood of the crack front (small compared to the thickness of an individual layer h_i), i.e., substantially by the chemical and physical properties of the interfacial surfaces of the layers in the continuation of the crack. It is these properties that explain the observable effects of post-critical delamination crack growth and the failure of multilayer structures under relatively low loads.

Three kinds of delamination cracks. Depending on the state of stress of the multilayer shell, we shall distinguish /2/ the following special cases near the crack front (Fig.2): normal rupture or separation $(N_n = 0,\ T_n = 0,\ M_n \neq 0)$, transverse shear $(M_n = 0,\ T_n = 0,\ N_n \neq 0)$, and longitudinal shear $(M_n = 0,\ N_n = 0,\ T_n \neq 0)$.

The stress and strain distribution near the crack front is found by solving the boundary value problem (1.8) and (1.9) for the boundary layer (Fig.2b). In the case of a linearly-elastic body this problem can be solved by the Wiener-Hopf method using a Fourier transform in n. The regularities of delamination crack development can be investigated directly by using the general equation (1.12) without analyzing the stresses and strains in the boundary layer itself.

2. Investigation of the laws of delamination crack development by tests.

Equation (1.12) enables the intimate and complex processes of fracture at the end of cracks in macrospecimens with cracks to be studied by measuring the changes in two macroquantities in a time t: the crack length l and the quantity $[U_s]$. The research scheme, in principle, can be clarified by the simplest example of normal rupture of a two-layer specimen by tip forces.

Suppose it is required to study the development of normal rupture delamination cracks propagating along the interfaces of two layers in a multilayer shell. To do this it is necessary to prepare two-layer beams of the material of these layers 1 and 2, by reproducing exactly the method and technology of connecting their surfaces as well as the external conditions. The dimensions of the beam-specimens can be arbitrary. Then the beam with the artificially produced initial delamination crack of length l is stretched by two transverse forces Q according to a given program $Q(t)$. In this case, we have at the crack tip according to (1.11) (taking curvature into account)

$$M_n^\circ = 0,\ \varkappa_n^\circ = 0,\ M_n^+ = -M_n^- = Q(t)\,l(t) \tag{2.1}$$
$$[U_s] = Ql(\varkappa_n^+ - \varkappa_n^-) - \int Ql\,d(\varkappa_n^+ - \varkappa_n^-)$$

The quantities $\varkappa_n^\pm(t)$ at the crack tip can be measured directly, but it is more convenient and exact to compute them from the rheological model of materials 1 and 2 determined from independent experiments on homogeneous materials without cracks. Therefore, using (2.1) the function $[U_s(t)]$ is found from the measured function $l(t)$. By Eq.(1.12) this equals $\Gamma(t)$, and the determination of this function is the purpose of the investigation.

The results of this study should be displayed in invariant variables in the form of a dependence of the rate of crack growth $l^\cdot$ on the quantity Γ; in the general case this dependence will also be determined by the loading described by the function $[U_s(t)]$.

Using the postulate of similitude /1/, the $l^{\cdot} - \Gamma$ diagram obtained can be used to compute the normal rupture delamination crack development between materials 1 and 2 in any arbitrarily loaded multilayer shell for the same function $[U_s(t)]$. To do this it is necessary to substitute the function $\Gamma(l^{\cdot})$ from the diagram into the left side of (1.12) and solve the differential equation obtained for $l(t)$. A sufficiently large set of $l^{\cdot} - \Gamma$ diagrams and their corresponding "loading paths" $[U_s(t)]$ can describe delamination crack development for any pair of materials with arbitrary properties. It is here more convenient to determine the sets $l(t)$ and $[U_s(t)]$ in the computations initially from the known diagrams, and to select the function $l(t)$ to be realized from the computed function that is closest to the "specified" function $[U_s(t)]$.

Computation using the critical state. The crack velocity $l^{\cdot}$ usually increases smoothly as Γ increases (at least it does not decrease). But a critical value $\Gamma = 2\gamma_{fm}$ exists where an abrupt increase in $l^{\cdot}$ is observed as it is approached. We will call the quantity γ_{fm} the specific adhesion energy /2/ of the pair of materials f and m. From the preceding it is clear that it depends, in general, on the function $[U_s(t)]$, i.e., on the prehistory. However, like the precritical crack growth, such a dependence can (or must) often be neglected for many materials and external conditions. The general equation (1.12) hence acquires the following simple form:

$$[U_s] = 2\gamma_{fm} \tag{2.2}$$

We will give several simple examples of a computation using the critical state (2.2).

3. Fracture of a two-layer beam of elastic-plastic materials.

Consider the fracture diagram of a two-layer beam by two cantilever forces. Each material is ideally elastic-plastic. The force Q is assumed to be non-decreasing (simple loading).

From the $\sigma_n - \varepsilon_n$ diagram the $M_n - \varkappa_n$ diagram can be computed for pure beam bending. Then we calculate from (1.11)

$$\Gamma = M_n\varkappa_n - \int_0^{\varkappa_n} M_n\,d\varkappa_n = \begin{cases} \frac{1}{24}Eh^3\varkappa_n^2 & \left(0 \leqslant \varkappa_n \leqslant \frac{2\sigma_s}{hE}\right) \\ \frac{1}{2}\frac{h}{E}\sigma_s^2 - \frac{2\sigma_s^3}{3E^2\varkappa_n} & \left(\frac{2\sigma_s}{hE} \leqslant \varkappa_n < \infty\right) \end{cases}$$

We now determine $\Gamma = U_s^+ + U_s^-$ $(U_s^\circ = 0)$ at the crack tip, where $M_n = Ql$. Depending on the relationship between the constants and the magnitudes of the loads, the following modifications are possible:

For $6Ql < \sigma_{s1}h_1^2 \leqslant \sigma_{s2}h_2^2$ (elastic layer 1 plus elastic layer 2)

$$\Gamma = 6Q^2l^2\left(\frac{1}{E_1h_1^3} + \frac{1}{E_2h_2^3}\right) \tag{3.1}$$

For $\sigma_{s1}h_1^2 < 6Ql < \sigma_{s2}h_2^2$ (elastic-plastic layer 1 plus elastic layer 2)

$$\Gamma = \frac{h_1\sigma_{s1}^2}{2E_1} - \frac{\sigma_{s1}^2}{\sqrt{3}E_1}\sqrt{h_1^2 - \frac{4Ql}{\sigma_{s1}}} + \frac{6Q^2l^2}{E_2h_2^3} \tag{3.2}$$

For $Ql = {}^1/_4\sigma_{s1}h_1^2 \leqslant {}^1/_6\sigma_{s2}h_2^2$ (elastic layer 2 plus elastic-plastic layer 1 in the limit state, with a plastic hinge at the crack tip)

$$\Gamma = \frac{h_1\sigma_{s1}^2}{2E_1} + \frac{3\sigma_{s1}^2h_1^4}{8E_2h_2^3} \tag{3.3}$$

For $\sigma_{s1}h_1^2 \leqslant \sigma_{s2}h_2^2 < 6Ql \leqslant {}^3/_2\sigma_{s1}h_1^2$ (elastic-plastic layer 1 plus elastic-plastic layer 2)

$$\Gamma = \frac{h_1\sigma_{s1}^2}{2E_1} + \frac{h_2\sigma_{s2}^2}{2E_2} - \frac{\sigma_{s1}^2}{\sqrt{3}E_1}\sqrt{h_1^2 - \frac{4Ql}{\sigma_{s1}}} - \frac{\sigma_{s2}^2}{\sqrt{3}E_2}\sqrt{h_2^2 - \frac{4Ql}{\sigma_{s2}}} \tag{3.4}$$

Formulas (3.1)-(3.4) enable the development of the fracture process in an elastic-plastic two-layer beam with a crack to be analyzed using the crack growth criterion (2.2). Depending on the relationship between the constants, the following modifications are possible (Q_b is the limit load):

For $6Q_bl < \sigma_{s1}h_1^2 \leqslant \sigma_{s2}h_2^2$ crack growth starts before the origination of plastic zones in layer 1 and 2 (quasi-brittle fracture)

$$Q_b = \frac{1}{l}\sqrt{\frac{1}{3}\gamma_{fm}}\left(\frac{1}{E_1h_1^3} + \frac{1}{E_2h_2^3}\right)^{-1/2} \tag{3.5}$$

For $\sigma_{s1}h_1^2 < 6lQ_b < \sigma_{s2}h_2^2$ crack growth starts after the origination of plastic zones in layer 1 (brittle-plastic layer 2 plus a quasi-brittle layer 1), while Q_b is the root of the equation

$$\frac{6Q_b{}^2l^2}{E_2h_2{}^3} - \frac{\sigma_{s1}{}^2}{\sqrt{3}E_1}\sqrt{h_1{}^2 - \frac{4Q_bl}{\sigma_{s1}}} = 2\gamma_{fm} - \frac{h_1\sigma_{s1}{}^2}{2E_1} \qquad (3.6)$$

For $\sigma_{s1}h^2 \leqslant \sigma_{s2}h_2{}^2 < 6Q_bl \leqslant {}^3/_2\,\sigma_{s1}h_1{}^2$ crack growth starts after development of the plastic zones in layers 1 and 2 (brittle-plastic layers 1 and 2), while Q_b is the root of the equation

$$\frac{\sigma_{s1}{}^2}{\sqrt{3}E_1}\sqrt{h_1{}^2 - \frac{4Q_bl}{\sigma_{s1}}} + \frac{\sigma_{s2}{}^2}{\sqrt{3}E_2}\sqrt{h_2{}^2 - \frac{4Q_bl}{\sigma_{s2}}} = \frac{h_1\sigma_{s1}{}^2}{2E_1} + \frac{h_2\sigma_{s2}{}^2}{2E_2} - 2\gamma_{fm} \qquad (3.7)$$

If the conditions

$$\frac{h_1\sigma_{s1}{}^2}{2E_1} + \frac{3\sigma_{s1}{}^2h_1{}^4}{8E_2h_2{}^3} < 2\gamma_{fm}, \quad \sigma_{s1}h_1{}^2 < \frac{2}{3}\sigma_{s2}h_2{}^2$$

or

$$\frac{h_1\sigma_{s1}{}^2}{2E_1} + \frac{h_2\sigma_{s2}{}^2}{2E_2} - \frac{\sigma_{s2}{}^2}{\sqrt{3}E_2}\sqrt{h_2{}^2 - \frac{\sigma_{s1}}{\sigma_{s2}}h_1{}^2} < 2\gamma_{fm}$$
$${}^3/_2\sigma_{s1}h_1{}^2 > \sigma_{s2}h_2{}^2 > \sigma_{s1}h_1{}^2$$

are satisfied, then the crack does not develop, and fracture occurs because of the depletion of the carrying capacity of beam 1 for the elastic or elastic-plastic state of the layer 2

$$Q_b = \sigma_{s1}h_1{}^2/4l \qquad (3.8)$$

Crack growth in the problem under consideration is always accelerating, and unstable since $Q_b \sim 1/l$.

In the case of rigidly plastic materials in both layers, when $\gamma_{fm}E_1 \gg h_1\sigma_{s1}{}^2, \gamma_{fm}E_2 \gg h_2\sigma_{s2}{}^2$, the crack does not develop, while fracture occurs because of the formation of a plastic hinge in beam 1 at the tip of the crack (see (3.8)). It can be shown that this result holds for the arbitrary case of loading multilayer shells with delamination cracks.

Transverse rupture crack in a two-layer beam. In this case we have

$$M_n{}^\circ = M_n{}^+ = M_n{}^- = 0,\ T_n{}^\circ = T_n{}^+ = T_n{}^- = 0 \qquad (3.9)$$
$$N_n{}^0 = 0,\ N_n{}^+ = -N,\ N_n{}^- = N$$
$$N = hE\varepsilon_n,\ U_s = {}^1/_2hE\varepsilon_n{}^2 \ \text{when}\ |\varepsilon_n| < \varepsilon_s$$
$$N = hE\varepsilon_s,\ U = {}^1/_2hE\varepsilon_s{}^2 \ \text{when}\ |\varepsilon_n| > \varepsilon_s$$

Hence, using (2.2) we find the limit loads N_b in the case of monotonic growth of N:

For $N_b < h_1\sigma_{s1} < h_2\sigma_{s2}$ (quasi-brittle fracture)

$$N_b = 2\sqrt{\gamma_{fm}}\left(\frac{1}{h_1E_1} + \frac{1}{h_2E_2}\right)^{-1/2}$$

and for $h_1\sigma_{s1} < 2\sqrt{\gamma_{fm}}\ (h_1{}^{-1}E_1{}^{-1} + h_2{}^{-1}E_2{}^{-1})^{-1/2}$ (plastic fracture of layer 1 with non-developing cracks)

$$N_b = h_1\sigma_{s1}$$

4. Subcritical delamination crack growth in multilayer shells of elastic-plastic materials. Consider the simple example of subcritical development of normal rupture delamination cracks in a two-layer beam under pure bending by a moment M. In this case the quantity Γ in the subcritical state is determined by (3.1),(3.2) and (3.4) for simple loading, where $Ql = M$ must be substituted. It is independent of the crack length l; as is easily verified, this independence of l is preserved for any arbitrarily complex loading paths $M(t)$ although the corresponding expressions (3.2) and (3.4) will already be different (they will depend on the preceding loading path).

By a direct computation it is easy to confirm that Γ will be less (or at least not greater) for a complex loading path than for a simple loading path. This holds not only for this example, but also in the general case of a delamination crack in multilayer shells from elastic-plastic materials. All the regularities of subcritical crack growth have the following general mature: crack development starts as soon as a certain threshold value of Γ, dependent on the specific physical mechanism of the fracture process directly at the crack front /1, 2/, is exceeded. Hence, the following general conclusion follows: if the external loads acting on the shell increase rapidly, and then decrease to normal working values, this will generally result in an increase in the threshold value of Γ. This physical effect of plasticity "in the large" can turn out to be useful in many practical applications.

It is seen that the Bauschinger effect results in a still greater decrease in $[U_s]$. Hence, a substantial increase in the threshold load level at which subcritical crack growth starts can be achieved by preliminary application of several "loading-unloading" cycles on the shell.

For certain cases we will find the velocity of subcritical delamination crack growth $l^\cdot$

in the case under consideration on the basis of (1.12) and the following general regularities of subcritical crack growth /1,2/:

crack growth following instantaneously after loading

$$l^{\cdot} = \frac{\beta_{fm}[U_s][U_s^{\cdot}]}{2\gamma_{fm}(2\gamma_{fm} - [U_s])} \quad \text{when} \quad [U_s^{\cdot}] \geqslant 0 \tag{4.1}$$

crack development because of hydrogen embrittlement (in metals)

$$l^{\cdot} = A_{fm}\,(\sqrt{[U_s]} - B_{fm}) \tag{4.2}$$

crack growth because of the kinetic local ageing mechanism

$$l^{\cdot} = v_{fm} \exp \frac{\alpha_{fm}\sqrt{[U_s]}}{RT} \tag{4.3}$$

and crack growth because of corrosion

$$l^{\cdot} = \text{const} \tag{4.4}$$

Evidently (4.1)-(4.4) are suitable even in the general case of delamination cracks in multilayer shells, where (4.1) must be supplemented by the condition of crack irreversibility under unloading ($l^{\cdot} = 0$ for $[U_s^{\cdot}] < 0$). Strictly speaking, (4.2)-(4.4) are obtained for quasistationary loads. Under the simultaneous action of several subcritical crack growth mechanisms, the corresponding increments to the crack length must be added for each mechanism if there is no mutual influence on each other. In the case of a transverse rupture crack the quantity $[U_s]$ is independent of l and the loading path. In the case of a cantilever loading by forces, the crack will be accelerated; the exact form of $l(t)$ can be found from the appropriate differential equation obtained after substituting $-[U_s]$, using (3.3)-(3.6), into equations of the type (4.1)-(4.4).

5. Dependence of the limit loads on the loading path.

Consider the simplest case of pure bending of a beam on a rigid substrate (when layer 2 in a two-layer beam is absolutely rigid). The dependence of the bending moment M_n on the curvature $\varkappa_n$ in the beam foundation (at the tip of the crack) is displayed in Fig.4. Using this diagram, the value of $[U_s]$ is easily evaluated for any loading path. For instance, we have

$[U_s] = S\,(OAE)$ at point A along path OA

$[U_s] = -S\,(OABC)$ at point C along path $OABC$

$[U_s] = S\,(OAE) - A_B\,(ABCD)$ at point A on the path $OABCD$,

etc. Here S is the area of the appropriate curvilinear figure in Fig.4, and A_B is the area of the Bauschinger hysteresis loop.

By generalizing these constructions it is possible to arrive at general conclusions.

A. The limit loads for multilayer elastic-plastic shells with delamination cracks are independent of the loading path if secondary effects of the Bauschinger type can be neglected.

B. The Bauschinger hysteresis effect displayed in Fig.4 results in an increase in the limit loads for multilayer shells with delamination cracks.

These deductions are generally valid for any "initiation" loads for which crack growth starts.

Therefore, when there is no Bauschinger effect, the quantity $[U_s]$ is a function of M_n, N_n, T_n, not explicitly time-dependent, and calculable over a simple loading path (this is equivalent to the assumption that a shell with a crack is non-linearly elastic). Therefore, in this case the loads for which delamination crack growth starts in multilayer shells can be calculated assuming appropriate non-linearly elastic behaviour of the material.

6. Fatigue delamination cracks.

Let the external loads be certain periodic functions of time. Then $[U_s]$ will also be a certain periodic function of time with weakly varying coefficients. Using the regularities of subcritical crack development, of the type (4.1)-(4.4), increments in the crack can be found for any programmed or random loading /2/. We will confine ourselves here solely to taking account of the plastic effects of the instantaneous reaction described by (4.1).

Integrating this equation, we can arrive at the following regularity by using a well-known approach /2/

$$\frac{dl}{dn} = -\beta_{fm}\left(\frac{[U_s^{\max}] - [U_s^{\min}] - A_B}{2\gamma_{fm}} - \ln\frac{2\gamma_{fm} - [U_s^{\max}] + \Delta S + A_B}{2\gamma_{fm} - [U_s^{\min}] + \Delta S}\right)$$

Here n is the number of load cycles (it plays the role of time), dl/dn is the crack velocity, $[U_s^{\max}]$ and $[U_s^{\min}]$ are the appropriate greatest and least values of $[U_s]$ during the cycle, calculated along a simple loading path, A_B is the area of the corresponding Bauschinger hysteresis loop, and ΔS is the component indicating the influence of the prehistory at the beginning of the cycle (in the majority of cases it can be set equal to zero, see Fig. 4).

Elastic multilayer shell with quasibrittle delamination cracks. In this case Γ is expressed in terms of the stress intensity factors /2/, hence (1.12) can be used to evaluate the intensity factors by means of the forces and bending moments determined from a computation of the elastic shell.

Optimal design of multilayer shells with delamination cracks. Experience shows that delamination crack formation in multilayer structures is inevitable, especially during use. It is consequently important that the damage produced should be safe and not endanger the structure. The analysis performed above enables us to extract the most important structural parameter of a multilayer shell that controls this process. This parameter is the quantity Γ; the structure should be designed so that this quantity is greatest in the most dangerous zones from the viewpoint of delamination . The technology for producing the structure should ensure the greatest possible deceleration of delamination cracks; the most important physical characteristics for the strength of the layer connection are the parameters $\gamma_{fm}, \beta_{fm}, v_{fm}, \alpha_{fm}, A_{fm}$, etc. Depending on the conditions of structure operation, any of these parameters can play the main role in selecting the optimal technology and the optimal multilayer material.

7. **Examples illustrating this theory.** *Elliptical crack in a plate.* Suppose a flat plate is bonded together from two identical layers of thickness h. There is a crack of elliptic planform L on the layer boundary: $x^2/a^2 + y^2/b^2 = 1$. A constant pressure p is applied to the crack edges. Because of symmetry it is sufficient to consider the strain of the upper layer.

We have for the normal displacement w of the layer

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{k_b} \tag{7.1}$$

$$L: w = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0 \quad \left(k_b = \frac{Eh^3}{12(1-\nu^2)}\right)$$

The solution of this problem has the following form:

$$w = A\frac{p}{k_b}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2, \quad A = \frac{1}{8}(3a^{-4} + 2a^{-2}b^{-2} + 3b^{-4})^{-1} \tag{7.2}$$

In this case Γ equals (we take account of the presence of the second layer)

$$\Gamma = k_b \varkappa_n^2 = 64A^2\frac{p^2}{k_b}\left[\left(\frac{x^2}{a^4} - \frac{y^2}{b^4}\right)^2 + \frac{4x^2y^2}{a^4b^4}\right] \tag{7.3}$$

$$((x, y) \in L)$$

We also find the stress intensity factor

$$K_I = \sqrt{\frac{E[U_s]}{1-\nu^2}} = 16Ap\frac{\sqrt{3}}{h^{3/2}}\left[\left(\frac{x^2}{a^4} - \frac{y^2}{b^4}\right)^2 + \frac{4x^2y^2}{a^4b^4}\right]^{1/2} \tag{7.4}$$

Therefore, an elliptic delamination crack always starts to develop along the minor axis; the crack outline changes until it becomes circular with a diameter equal to the major diameter $2b$ of the initial ellipse. Afterwards, the crack grows along the whole outline while remaining circular.

Let us estimate crack growth when there is an increase in pressure, using the following approximate assumption: the crack outline always remains elliptical with a minor diameter equal to $2b = 2b(p)$ and an invariant major diameter $2a$. We will confine ourselves to considering only the limit states when $K_{I\max} = K_{Ic}$ at the point $x = 0$, $y = \pm b$; subcritical crack development can be studied by using this same assumption.

From (7.4) we obtain

$$\bar{p} = 3\bar{b}^2 + 2 + 3\bar{b}^{-2} \tag{7.5}$$

$$(\bar{p} = 2\sqrt{3}a^2ph^{-3/2}K_{Ic}^{-1}, \ \bar{b} = b/a \leqslant 1)$$

As is seen, the crack develops unstably on reaching the limit pressure p_b corresponding to the initial quantity b_0, changing to a circular shape when $b = 1$. In the case of a disk-shaped crack

$$K_I = \frac{\sqrt{3}pr^2}{4h^{3/2}}, \quad \bar{p} = \frac{8}{\bar{r}^2} \quad \left(\bar{r} = \frac{r}{a}\right)$$

We note the limiting case $b/a \to 0$ (the strip $|y| < b$). In this case, by (7.4)

$$K_I = \frac{2pb^2}{h\sqrt{3h}}, \quad \bar{p} = \frac{3}{\bar{b}^2}\left(\bar{b} = \frac{b}{h}\right)$$

Unlike the case of an elliptical crack in a three-dimensional body, an elliptical delamination crack in a plate has no section of stable growth (however, such sections should, obviously, appear when there are other layers of more rigid material).

A parabolic crack in a plate. Under the conditions of the preceding problem, let a delamination crack occupy a domain $y \geqslant x^2/(2r_0)$ in planform, where r_0 is the radius of curvature of the parabola at its apex. The solution of the boundary value problem (7.1) for this domain is:

$$w = \frac{pr_0^2}{6k_b}\left(y - \frac{x^2}{2r_0}\right)^2 \tag{7.6}$$

$$\Gamma = k_b \varkappa_n^2 = \frac{p^2 r_0^2}{9k_b}\left[(x^2 - r_0^2)^2 + 4x^2\right]$$

$$K_I = \frac{2r_0 p_0}{h\sqrt{3h}}\sqrt{(x^2 - r_0^2)^2 + 4x^2}$$

The function $K_I(x)$ has one local maximum for $x = 0$ and two absolute minima for $x = \pm r_0/\sqrt{5}$.

A hyperbolic crack in a plate. Under the conditions of the preceding problems, let the crack occupy a domain $x^2/a^2 - y^2/b^2 \leqslant 1$ in planform, where a and b are the parameters of the hyperbola.

The solution of problem (7.1) for this domain is

$$w = A\frac{p}{k_b}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)^2 \tag{7.7}$$

$$\Gamma = 64A^2\frac{p^2}{k_b}\left[\left(\frac{x^2}{a^4} - \frac{y^2}{b^4}\right)^2 + \frac{4x^2y^2}{a^4b^4}\right]$$

$$A = \frac{1}{8}(3a^{-4} - 2a^{-2}b^{-2} + 3b^{-4})^{-1}$$

Delamination crack in a two-layer cylindrical shell. In a circular cylindrical shell of radius R and thickness $2h$ let there be an axisymmetric delamination crack of length $2L$ which separates the initial shell into two cylindrical shells of thickness h_1 and h_2, made from different materials $(2h = h_1 + h_2)$. A constant pressure p is applied to the crack edges.

The radial displacement w of cylindrical shells is determined from the boundary value problem

$$\frac{d^4w}{dx^4} + \frac{Eh_i}{R^2k_i}w = \frac{p}{k_i}, \quad k_i = \frac{E_ih_i^3}{12(1-\nu_i^2)} \quad (i = 1, 2) \tag{7.8}$$

$$x = \pm L, \quad w = 0, \quad dw/dx = 0$$

We write the solution of the boundary value problem (7.8) thus:

$$w_i = \frac{pR^2}{E_ih_i}\left(1 - \frac{\sin\lambda_iL\,\mathrm{ch}\,\lambda_ix + \mathrm{sh}\,\lambda_iL\cos\lambda_ix}{\sin\lambda_iL\,\mathrm{ch}\,\lambda_iL + \mathrm{sh}\,\lambda_iL\cos\lambda_iL}\right) \tag{7.9}$$

$$\lambda_i = \left(\frac{E_ih_i}{k_iR^2}\right)^{1/4} = 12\left(\frac{1-\nu_i^2}{R^2h_i^2}\right)^{1/4} \quad (i = 1,2)$$

We hence determine

$$\Gamma = \frac{1}{2}k_{b1}\varkappa_{n1}^2 + \frac{1}{2}k_{b2}\varkappa_{n2}^2 = \frac{1}{2}p^2R^2\sum_{i=1}^{2}\frac{1}{E_ih_i}\left(\frac{\mathrm{tg}\,\lambda_iL - \mathrm{th}\,\lambda_iL}{\mathrm{tg}\,\lambda_iL - \mathrm{th}\,\lambda_iL}\right)^2 \tag{7.10}$$

This formula enables us to investigate the critical and subcritical development of a delamination crack. We will study only the critical state in which the criterion $\Gamma = 2\gamma_{fm}$ is satisfied; we will combine ourselves to the case of two identical layers of thickness $h_1 = h_2 = h$, $\lambda_1 = \lambda_2 = \lambda$. According to (7.10), we have in this case

$$\bar{p} = \frac{\mathrm{tg}\,\bar{L} + \mathrm{th}\,\bar{L}}{\mathrm{tg}\,\bar{L} - \mathrm{th}\,\bar{L}} \quad \left(\bar{p} = \frac{Rp_b}{\sqrt{2Eh\gamma_{fm}}}, \quad \bar{L} = \lambda L\right) \tag{7.11}$$

A graph of the function $\bar{p}(\bar{L})$ is displayed in Fig.5. It consists of an infinite number of "almost-periodic" unstable branches having the lines

$$\lambda L_b = 0, \quad \frac{5}{4}\pi - 4\cdot 10^{-4}, \quad \frac{9}{4}\pi, \quad \frac{13}{4}\pi, \quad \frac{17}{4}\pi, \ldots$$

as their asymptotes.

The behaviour of the delamination crack in a cylindrical shell turns out to be very peculiar . For any initial crack L_0 it will always grow to the nearest value L_b to the right, and will remain in this position $L = L_b$ for as long as desired and under any loads (since the quantity Γ is zero for $L = L_b$). Crack growth in the critical state will be unstable, rapid, and may be fairly slow in the subcritical state. The presence of an infinite number of possible "barrier" states in which the crack is "stationary" and not subjected to any growth in the critical or subcritical regimes relates the system considered to quantum-mechanical systems for which such behaviour is typical.

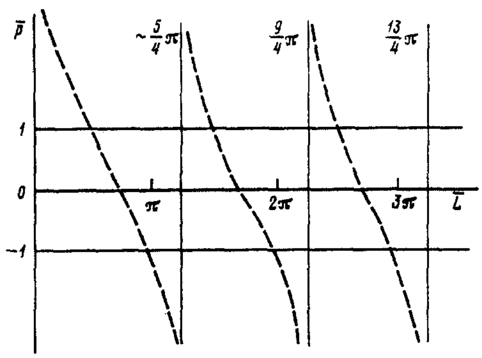

Fig.5

An elliptical crack in a membrane. It was assumed in the examples considered above that the bending stresses are large compared with the tensile stresses of the middle surface. We will consider another extreme case when the bending stresses are small compared with the preliminary tension of the plate by stresses σ_x and σ_y.

Initially we note the simple general formula for $[U_s]$ in the case of a delamination crack in an arbitrary multilayer membrane shell

$$[U_s] = \frac{1}{2}\mu^+\left(\frac{\partial w^+}{\partial n}\right)^2 + \frac{1}{2}\mu^-\left(\frac{\partial w^-}{\partial n}\right)^2 \qquad (7.12)$$
$$(\mu^\pm = \sum h_i^\pm \sigma_{ni}^\pm)$$

Therefore, by (1.12) and (7.12), delamination crack development in a membrane shell is determined by the angles of rotation of the corresponding shells at the point of the crack outline under consideration.

Let a plane elastic membrane of thickness $2h$ be bonded together from two identical layers; there is an elliptical delamination crack L on the interface, to whose edges a constant pressure p is applied.

We have for the normal displacement w of the upper layer

$$\sigma_x \frac{\partial^2 w}{\partial x^2} + \sigma_y \frac{\partial^2 w}{\partial y^2} = -\frac{p}{h} \qquad (7.13)$$
$$w = 0 \text{ when } (x, y) \in L$$

The solution of this boundary value problem has the form

$$w = D\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \quad D = \frac{p}{2h}\left(\frac{\sigma_x}{a^2} + \frac{\sigma_y}{b^2}\right)^{-1} \qquad (7.14)$$
$$\Gamma = h\sigma_n\left(\frac{\partial w}{\partial n}\right)^2 = 4hD^2\left(\sigma_x \frac{x^2}{a^4} + \sigma_y \frac{y^2}{b^4}\right)$$

As is seen, for $a^2\sigma_y > b^2\sigma_x$ the crack starts to develop along the y axis, while for $a^2\sigma_y < b^2\sigma_x$ it develops along the x axis. During development, for given σ_x and σ_y the crack shape changes, tending to an ellipse $x^2/\sigma_x + y^2/\sigma_y = \lambda$ describing the initial shape; later the crack development reduces to a self similar expansion of this last ellipse because of monotonic growth in the parameter λ ($\Gamma =$ const along the contour on this ellipse). For fixed σ_x and σ_y the crack growth is always unstable as p increases. The parameters σ_x and σ_y are stabilizing factors, hence, depending on the loading path in the three-dimensional space (p, σ_x, σ_y) patterns of any change in the elliptic cracks (neutral equilibrium, stability, instability, "snaps", etc.) can be obtained.

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